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The lower central series of subgroups of the Vershik–Kerov group

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ABSTRACT

We describe the derived and the lower central series of certain subgroups of the Vershik–Kerov group. We are concerned with the subgroups consisting of infinite matrices having finite number of nonzero entries in each row. We consider the group of matrices over rings which are associative, commutative, of stable rank at most one and such that the identity can be written as a sum of two units. For this case we give a complete description of the derived and the lower central series. Moreover, we prove that every element of discussed commutator subgroups can be written as a product of a finite number of commutators.

1. Introduction

For any associative ring R with 1, by $\mathrm{GL}_{Cf,\infty}(R)$ we denote the group of invertible infinite (i.e. $\mathbb{N} \times \mathbb{N}$) matrices, which are column-finite and whose inverses also have this property. In this paper we are concerned with the Vershik–Kerov group. This group, denoted by $\mathrm{GL}_{VK,\infty}(R)$, is the subgroup of $\mathrm{GL}_{Cf,\infty}(R)$ consisting of matrices having only a finite number of nonzero entries under the main diagonal. The origin of the group $\mathrm{GL}_{VK,\infty}(R)$ (see [1,2]) is related to the asymptotic representation theory. This theory connects functional analysis, algebra, combinatorics and probability theory (for more details see e.g. [3]) and it concerns the classical groups of large or infinite dimensions.

It is known that for any commutative ring with 1, the commutator subgroup of $\mathrm{GL}_n(R)$ coincides with $\mathrm{E}_n(R)$ – the group generated by elementary transvections (i.e. matrices derived by taking the identity matrix and replacing one of the zero elements with a nonzero value). It is an interesting problem whether an analogous statement holds for the Vershik–Kerov group. For this reason we introduce the

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groups $\mathrm{SL}_{VK,\infty}(n,R)$, which consist of infinite matrices of the form

$$\left(\begin{array}{c|c} g & h \\ \hline 0 & k \end{array}\right),$$

where $g \in \mathrm{SL}_n(R)$ and k is an infinite unitriangular matrix. The union $\cup_{n=1}^{\infty} \mathrm{SL}_{VK,\infty}(n,R)$ will be denoted by $\mathrm{SL}_{VK,\infty}(R)$.

Does the commutator subgroup of the group $\mathrm{GL}_{VK,\infty}(R)$ *coincide with the group* $\mathrm{SL}_{VK,\infty}(R)$*?*

The problem cited above, posed by Sushchanskiĭ [4], as far as we are concerned, stays open. However, we are able to solve it in some special case.

We denote by $\mathrm{T}_\infty(R)$ the group of all upper triangular matrices, whose elements on the main diagonal are invertible in R, and by $\mathrm{UT}_\infty(R)$ the subgroup of $\mathrm{T}_\infty(R)$ consisting of all upper unitriangular matrices. The $\mathrm{GL}_{RB,\infty}(R)$ is the subgroup of $\mathrm{GL}_{VK,\infty}(R)$ containing all matrices of the form

$$\left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right),$$

where $g \in \mathrm{GL}_n(R)$ for some $n \in \mathbb{N}$ and e is the infinite identity matrix. Analogously, $\mathrm{SL}_{RB,\infty}(R)$ contains matrices of the same form, but with $g \in \mathrm{SL}_n(R)$.

From the equality

$$\left(\begin{array}{c|c} g & h \\ \hline 0 & k \end{array}\right) = \left(\begin{array}{c|c} e_n & 0 \\ \hline 0 & k \end{array}\right) \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right) \tag{1}$$

it is easy to conclude that it holds:

$$\mathrm{GL}_{VK,\infty}(R) = \mathrm{T}_\infty(R) \cdot \mathrm{GL}_{RB,\infty}(R).$$

Moreover, the group $\mathrm{GL}_{RB,\infty}(R)$ is a normal subgroup of $\mathrm{GL}_{VK,\infty}(R)$. However, since intersection $\mathrm{T}_\infty(R) \cap \mathrm{GL}_{RB,\infty}(R)$ is nontrivial, $\mathrm{GL}_{VK,\infty}(R)$ is not a semidirect product of these two subgroups. Nevertheless, this decomposition makes our task much easier. We restrict our attention to the subset (which indeed forms a subgroup) in which all matrices are of the form (1) and k is row-finite.

We also assume that R is a commutative, associative ring with 1, of stable rank at most 1. Moreover, we are concerned with R containing such θ that θ and $1-\theta$ are in R^*.

The first theorem we present is:

Theorem 1.1. *The commutator subgroup of* $\mathrm{GL}_{RB,\infty}(R)$ *coincides with the group* $\mathrm{SL}_{RB,\infty}(R)$ *if and only if the commutator subgroup of* $\mathrm{GL}_n(R)$ *coincides with* $\mathrm{SL}_n(R)$ *for all n.*

We are not only interested in $\mathrm{GL}'_{RB,\infty}(R)$, but also in its commutator width. We will show:

Theorem 1.2. *Commutator width of* $\mathrm{GL}_{RB,\infty}(R)$ *is less or equal to* 3.

We will also deal with $\mathrm{T}_{Rf,\infty}(R)$ and $\mathrm{UT}_{Rf,\infty}(R)$ – the groups of triangular and unitriangular row-finite matrices. We will prove:

Theorem 1.3. *The commutator subgroup of* $\mathrm{T}_{Rf,\infty}(R)$ *coincides with the group* $\mathrm{UT}_{Rf,\infty}(R)$.
The subgroup $[\mathrm{T}_{Rf,\infty}(R), \mathrm{UT}_{Rf,\infty}(R)]$ *coincides with* $\mathrm{UT}_{Rf,\infty}(R)$.

Theorem 1.4. *Commutator width of* $\mathrm{T}_{Rf,\infty}(R)$ *is less or equal to* 4.

2. Notation and basic facts

We denote by e_n the identity matrix of dimension n and by e the infinite identity matrix. By $\mathbf{0}$ we denote the matrix consisting only of zeros, regardless of its dimension. By $\mathrm{diag}(a_1, a_2, a_3, \ldots)$ we understand the diagonal matrix with elements $a_1, a_2, a_3, \ldots$ on the main diagonal and by $\mathrm{Diag}(A_1, A_2, A_3, \ldots)$ the block-diagonal matrix with blocks $A_1, A_2, A_3, \ldots$ on the diagonal. By $(h)_i$ we understand the ith row of a matrix h.

The following definitions of $G^{(n)}$ and $\gamma_n(G)$ are used along the paper (see e.g. [5]):

$$G^{(0)} = G, \qquad G^{(n+1)} = [G^{(n)}, G^{(n)}] \quad \text{for } n \geqslant 0,$$

$$\gamma_0(G) = G, \qquad \gamma_{n+1}(G) = [\gamma_n(G), G] \quad \text{for } n \geqslant 0.$$

We denote by $\mathrm{UT}_n^i(R)$ the subgroup of $\mathrm{UT}_n(R)$ which contains all matrices u satisfying $u_{kl} = 0$ for $0 < l - k \leqslant i$.

The following lemma will be needed.

Lemma 2.1. *If R is a commutative, associative ring with 1, of stable rank less or equal to 1, and contains such θ that $\theta, 1 - \theta \in R^*$, then:*

1. $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] = \mathrm{SL}_n(R)$,
2. $[\mathrm{SL}_n(R), \mathrm{SL}_n(R)] = \mathrm{SL}_n(R)$,
3. $[\mathrm{T}_n(R), \mathrm{T}_n(R)] = \mathrm{UT}_n(R)$,
4. $\gamma_i(\mathrm{UT}_n(R)) = \mathrm{UT}_n^{i-1}(R)$,
5. $(\mathrm{UT}_n(R))^{(i)} = \mathrm{UT}_n^{2^i-1}(R)$.

Some of these facts are well known (see e.g. [5]). Some basic information may be found in classical textbooks (e.g. [6]).

Unless stated otherwise, we assume that R is as in the lemma given above.

The groups $\mathrm{GL}_{VK,\infty}(n, R)$ and $\mathrm{SL}_{VK,\infty}(n, R)$ are given by

$$\mathrm{GL}_{VK,\infty}(n, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & k \end{array} \right) \,\middle|\, g \in \mathrm{GL}_n(R),\ k \in \mathrm{T}_{Rf,\infty}(R) \right\}$$

$$\mathrm{SL}_{VK,\infty}(n, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & k \end{array} \right) \,\middle|\, g \in \mathrm{SL}_n(R),\ k \in \mathrm{UT}_{Rf,\infty}(R) \right\}$$

where $\mathrm{T}_{Rf,\infty}(R)$ is the subgroup of all row-finite triangular matrices, $\mathrm{UT}_{Rf,\infty}(R)$ – the subgroup of all unitriangular matrices which are also row-finite. Moreover, by $\mathrm{UT}_{Rf,\infty}(i, R)$ we denote the subgroup of $\mathrm{UT}_{Rf,\infty}(R)$ consisting of all matrices u satisfying $u_{kl} = 0$ for $0 < l - k \leqslant i$ (this is an analogue of $\mathrm{UT}_n^i(R)$). We also put:

$$\mathrm{GL}_{RB,\infty}(n, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array} \right) \,\middle|\, g \in \mathrm{GL}_n(R) \right\}$$

$$\mathrm{SL}_{RB,\infty}(n, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array} \right) \,\middle|\, g \in \mathrm{SL}_n(R) \right\}$$

$$\mathrm{T}_{RB,\infty}(n, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array} \right) \,\middle|\, g \in \mathrm{T}_n(R) \right\}$$

$$\left[\begin{array}{cccccccc|ccccc}
1 & \overbrace{0\ \cdots\ 0}^{k} & & * & \cdots & * & & & \multirow{7}{*}{$\left(\begin{array}{c} * \\ * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{array}\right)\begin{array}{c} \\ \\ \\ \\ \uparrow \\ k \\ \downarrow \end{array}$} & & & & \\
 & 1 & 0 & \cdots & 0 & * & \vdots & & & & & & \\
 & & 1 & 0 & & & * & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & \ddots & & & & & & & & \\
 & & & & & 1 & 0 & & & & & & \\
 & & & & & 0 & 1 & & & & & & \\
\hline
 & & & & & & & & 1 & & & & \\
 & & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & \ddots
\end{array}\right]$$

Fig. 1. $\mathrm{UT}_{RB,\infty}(n, k, R)$.

$$\mathrm{UT}_{RB,\infty}(n, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array} \right) \,\Big|\, g \in \mathrm{UT}_n(R) \right\}$$

$$\mathrm{UT}_{RB,\infty}(n, k, R) := \left\{ \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array} \right) \,\Big|\, g \in \mathrm{UT}_n^k(R),\ j > n - k \ \Rightarrow\ (h)_j = \mathbf{0} \right\}$$

The group $\mathrm{UT}_{RB,\infty}(n, k, R)$ is depicted in Fig. 1.

3. Commutator width of matrix groups

By $c(G)$ we understand the least c such that every element of the derived group $[G, G]$ is a product of at most c commutators.

It is known that in general (i.e. for arbitrary R) $c(\mathrm{SL}_n(R))$ does not have to be bounded. The classical results here are (see [7,8], respectively):

Theorem 3.1. *Let R be a commutative associative ring with 1, of stable rank less or equal to 1. Then $c(\mathrm{GL}_n(R)) \leqslant c(\mathrm{SL}_n(R)) \leqslant 5$.*

Theorem 3.2. *Let R be a commutative ring of stable rank less or equal to 1. Assume that either $n \geqslant 3$ or $n = 2$ and the identity is the sum of two units. Then $c(\mathrm{GL}_n(R)) \leqslant 2$.*

More information can be found in [9,10]. We will make use of the latter theorem.

Let us also cite a theorem, which is due to Dennis and Vaserstein (for more details see [7]):

Lemma 3.1. *For any associative ring R with 1 and any integer $n \geqslant 3$, every triangular matrix in $\mathrm{E}_n(R)$ – the group generated by elementary transvections, is the product of at most two commutators in $\mathrm{E}_n(R)$.*

The proof involves conjugation by a permutation matrix, so since we would like to obtain some upper bound for $c(\mathrm{T}_n(R))$, it can not be used by us. However, using a remark made there, we are able to show:

Lemma 3.2. *If R is an associative, commutative ring with 1 of stable rank at most 1, containing θ such that $\theta, 1 - \theta \in R^*$, then $c(\mathrm{T}_n(R)) \leqslant 2$.*

Proof. At first we show that we can write an unitriangular matrix with arbitrary second diagonal as a commutator of two triangular matrices. Let the second diagonal of a matrix $A \in \mathrm{UT}_n(R)$ consists of elements $a_1, a_2, \ldots, a_{n-1}$ and θ be such that $\theta,\ 1-\theta \in R^*$. Now we put $B = [X, D]$, where:

$$X = \begin{pmatrix} 1 & a_1(\theta^{-1}-1)^{-1} & & & & \\ & 1 & a_2(\theta-1)^{-1} & & & \\ & & 1 & a_3(\theta^{-1}-1)^{-1} & & \\ & & & 1 & a_4(\theta-1)^{-1} & \\ & & & & \ddots & \end{pmatrix}, \quad D = \begin{pmatrix} \theta & & & & \\ & 1 & & & \\ & & \theta & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}.$$

To be more precise, $X = e_n - \sum_{i=1}^{n-1} a_i(1-\theta)^{-1}(-\theta)^{\mathrm{imod}2}$, $D = \sum_{i=1}^{n} \theta^{\mathrm{imod}2}$.

Then the second diagonal of B consists of elements $a_1, a_2, \ldots, a_{n-1}$ as well.

Now, we make use of a remark made by the authors of [7] in the proof of Theorem 3.1:

> Every matrix $C \in UT_n(R)$ such that $C_{i,i+1} = 0$ for all $1 \leqslant i \leqslant n-1$, can be written as a commutator of Jordan matrix J and some triangular matrix T.

The matrix J from the cited claim is equal to $e_n + \sum_{i=1}^{n-1} e_{i,i+1}$.

Let A be an arbitrary unitriangular matrix. Then we define B (depending on A) to be equal to $[X, D]$, as in first part of the preceding proof and write $A = BC$. As $C = B^{-1}A$ and $A_{i,i+1} = B_{i,i+1}$ we have $C_{i,i+1} = 0$. Hence C can be written as a commutator $[T, J]$. Consequently, A is a product of at most two commutators.

Take notice that the obtained above results may be also applied to $\mathrm{UT}_n^1(R)$ – every matrix can be written as a commutator.

It is worth mentioning that if we restrict to the infinite fields, we can obtain even more (see [11]):

Theorem 3.3. *If K is an infinite field, then $c(\mathrm{T}_n(K)) = 1$ for $n \geqslant 1$.*

Now we prove:

Lemma 3.3. *Every matrix $A \in \mathrm{UT}_n^m(R)$, $m \geqslant 1$, can be written as a commutator of a Jordan matrix J and a matrix $T \in \mathrm{UT}_n^{m-1}(R)$.*

Proof. We begin with observing that if $A \in \mathrm{UT}_n^m(R)$, then the second diagonal of $\tilde{A} = JA$, where $J = e_n + \sum_{i=1} n - 1e_{i,i+1}$, contains only 1 and all entries on the next $m-1$ diagonals are 0. In fact, if indices u, v satisfy $1 \leqslant u \leqslant n-2$, $u+2 \leqslant v \leqslant u+m$, then $\tilde{A}_{uv} = J_{uu}A_{uv} + J_{uu+1}A_{u+1v} = 0$. From [7] we know that such $\tilde{A}$ is conjugated to J by some triangular matrix T whose entries satisfy:

$$T_{k,l-1} = T_{k+1,l} + \sum_{i>2} \tilde{A}_{k,k+i} T_{k-i,l} \tag{2}$$

for $l-k \geqslant 2$. This system can be solved inductively by putting $l-k = 2, 3, \ldots$ and setting arbitrarily some first entries of each diagonal. If we assume now that $A_{i,i+1} = \cdots = A_{i,i+m} = 0$, the system reduces to:

$$T_{k,l-1} = T_{k+1,l} + \sum_{i>m} \tilde{A}_{k,k+i} T_{k-i,l}. \tag{3}$$

Notice that for $l-k \leqslant m-1$ the equations of system (3) are of the form $T_{k,l-1} = T_{k+1,l}$, so we can assume that all those coefficients are equal to zero. The remained coefficients of T can be found using (3). Hence $A = [J, T]$, where $T \in \mathrm{UT}_n^{m-1}(R)$.

We can prove analogously:

Lemma 3.4. *Every matrix* $A \in \mathrm{UT}_n^{2^m-1}(R)$ *can be written as a commutator of* $J_{2^{m-1}-1} := e_n + \sum_{i=1}^{n-2^{m-1}} e_{i,i+2^{m-1}}$ *and some* $T \in \mathrm{UT}_n^{2^{m-1}-1}(R)$.

Proof. Let $J_{2^{m-1}-1}$ be as in the lemma and $A \in \mathrm{UT}_n^{2^m-1}(R)$. We put $\tilde{A} = J_{2^{m-1}-1}A$. Again, we want to show that $J_{2^{m-1}-1}$ and $\tilde{A}$ are conjugated by some T. This leads to the system similar to (2):

$$T_{k,l-2^{m-1}} = T_{k+2^{m-1},l-2^{m-1}} + \tilde{A}_{kl} + \sum_{i-k\geqslant 2^m,\, l-i\geqslant 2^{m-1}} \tilde{A}_{ki}T_{il}.$$

This time we can put $T_{kl} = 0$ for $l-k < 2^{m-1}$ and set other entries according to the above equations.

4. The derived and lower central series of the groups $\mathrm{GL}_{RB,\infty}(n, R)$ and $\mathrm{SL}_{RB,\infty}(n, R)$

Since we are going to deal with commutator subgroups, we begin with simple calculations. If we put:

$$m_1 = \left(\begin{array}{c|c} g_1 & h' \\ \hline 0 & e \end{array}\right), \quad m_2 = \left(\begin{array}{c|c} g_2 & h'' \\ \hline 0 & e \end{array}\right), \tag{4a}$$

then we have:

$$[m_1, m_2] = \left(\begin{array}{c|c} [g_1, g_2] & h \\ \hline 0 & e \end{array}\right), \tag{4b}$$

where

$$h = g_1^{-1}(g_2^{-1} - e_n)h' + g_1^{-1}g_2^{-1}(g_1 - e_n)h''. \tag{4c}$$

The notation as above will be used along this section.

We present now:

Theorem 4.1. *It holds:*

1. *The derived group of* $\mathrm{GL}_{RB,\infty}(n, R)$ *is* $\mathrm{SL}_{RB,\infty}(n, R)$.
2. *The derived group of* $\mathrm{SL}_{RB,\infty}(n, R)$ *is* $\mathrm{SL}_{RB,\infty}(n, R)$.
3. *We have* $[\mathrm{SL}_{RB,\infty}(n, R), \mathrm{GL}_{RB,\infty}(n, R)] = \mathrm{SL}_{RB,\infty}(n, R)$.

Proof. From (4b) it follows easily that $\mathrm{GL}'_{RB,\infty}(n, R) \leqslant \mathrm{SL}_{RB,\infty}(n, R)$.

Let $s = \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right) \in \mathrm{SL}_{RB,\infty}(n, R)$. One can see that g may be arbitrary. What we need to show is that h also may be arbitrary. This problem is equivalent to finding such g_1, g_2, h', h'' that:

$$(g_1 - e_n)h'' + (e_n - g_2)h' = g_2g_1h.$$

Let m_1 and m_2 be as in equality (4a). We put:

$$g_1 = \begin{pmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ & 1 & 1 & 0 & \dots & 0 \\ & & & & & \\ & & & 1 & 1 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & & & \\ 0 & 0 & & 1 & \\ -1 & 0 & & 0 & 1 \end{pmatrix},$$

$$h' = \begin{pmatrix} (g_2g_1h)_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad h'' = \begin{pmatrix} 0 \\ (g_2g_1h)_1 \\ (g_2g_1h)_2 \\ \vdots \\ (g_2g_1h)_{n-1} \end{pmatrix}.$$

It might be noticed that $[g_1, g_2] \neq e_n$. However, since $[g_1, g_2]^{-1}g$ is also in $\mathrm{SL}_n(R)$, it is possible to obtain any g and consequently – arbitrary s.

Since

$$\left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right)$$

may be written as

$$\left(\begin{array}{c|c} e_n & h \\ \hline 0 & e \end{array}\right)\left(\begin{array}{c|c} g & 0 \\ \hline 0 & e \end{array}\right),$$

the result follows.

One can see that when considering $[\mathrm{SL}_{RB,\infty}(n, R), \mathrm{SL}_{RB,\infty}(n, R)]$, in product (4b) we can obtain arbitrary $g \in \mathrm{SL}_n(R)$. We use the presented above g_1, g_2, h', h'' to obtain the desired result.

The last claim follows from $[\mathrm{GL}_n(R), \mathrm{SL}_n(R)] = \mathrm{SL}_n(R)$ and remarks in the proof of the first one.

Proof of Theorem 1.1. If the commutator subgroup of $\mathrm{GL}_n(R)$ coincides with $\mathrm{SL}_n(R)$ for all n, then by Theorem 4.1 we have $\mathrm{GL}'_{RB,\infty}(n, R) = \mathrm{SL}_{RB,\infty}(n, R)$ for all n. Hence $\mathrm{GL}'_{RB,\infty}(R) = \mathrm{SL}_{RB,\infty}(R)$. Suppose now that $\mathrm{GL}'_{RB,\infty}(R) = \mathrm{SL}_{RB,\infty}(R)$. From this it follows that for all n and for all $g \in \mathrm{SL}_n(R)$, the matrix $m = \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right)$ is in the commutator subgroup of $\mathrm{GL}_{RB,\infty}(R)$. Then by formulae (4a)–(4c) we have $g \in \mathrm{GL}'_n(R)$, so $\mathrm{SL}_n(R) \leqslant \mathrm{GL}'_n(R)$ for all n. Since $\mathrm{GL}'_n(R) \leqslant \mathrm{SL}_n(R)$, the claim follows.

Proof of Theorem 1.2. To prove that we have $c(\mathrm{GL}_{RB,\infty}(R)) == c(\cup_n \mathrm{GL}_{RB,\infty}(n, R)) \leqslant 3$ we use Theorem 3.2 and remarks from the proof of Theorem 4.1. From Theorem 3.2 we have $c(\mathrm{GL}_n(R)) \leqslant 2$ and from Theorem 4.1 we know that to obtain an arbitrary h in $\left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right) \in \mathrm{GL}'_{RB,\infty}(n, R)$ we need one more commutator.

Theorem 4.2. *We have:*

1. $[\mathrm{T}_{RB,\infty}(n, R), \mathrm{T}_{RB,\infty}(n, R)] = \mathrm{UT}_{RB,\infty}(n, R)$,
2. $[\mathrm{UT}_{RB,\infty}(n, R), \mathrm{T}_{RB,\infty}(n, R)] = \mathrm{UT}_{RB,\infty}(n, R)$.

Proof. Again, the inclusions $\mathrm{T}'_{RB,\infty}(n, R) \leqslant \mathrm{UT}_{RB,\infty}(n, R)$ and $\gamma_2(\mathrm{T}_{RB,\infty}(n, R)) \leqslant \mathrm{UT}_{RB,\infty}(n, R)$ are obvious. To show the reverse ones, it suffices to take:

$$g_1 = e_n, \quad g_2 = \begin{pmatrix} 1-\theta & & & \\ & 1-\theta & & \\ & & 1-\theta & \\ & & & \ddots \end{pmatrix}, \quad h' = \begin{pmatrix} \theta^{-1}(g_2g_1h)_1 \\ \theta^{-1}(g_2g_1h)_2 \\ \vdots \\ \theta^{-1}(g_2g_1h)_n \end{pmatrix}, \quad h'' = \mathbf{0},$$

where $\theta, 1 - \theta \in R^*$.

Now we will examine the series of $\mathrm{UT}_{RB,\infty}(n, R)$. We will prove the following.

Theorem 4.3. *It holds:*

1. $\gamma_k(\mathrm{UT}_{RB,\infty}(n, R)) = \mathrm{UT}_{RB,\infty}(n, k, R)$.
2. $\mathrm{UT}^{(k)}_{RB,\infty}(n, R) = \mathrm{UT}_{RB,\infty}(n, 2^k - 1, R)$.

Proof. We prove by induction. Let $k = 1$. It is clear that for all matrices $m_1, m_2 \in \mathrm{UT}_{RB,\infty}(n, R)$ we have $[g_1, g_2] \in \mathrm{UT}^1_n(R)$. On the other hand, every $g \in \mathrm{UT}^1_n(R)$ can be written as a commutator.

Let us consider the matrix h in $\left(\begin{array}{c|c} [g_1, g_2] & h \\ \hline 0 & e \end{array}\right)$ (as in formulae (4b) and (4c)). Since $g_1, g_2 \in \mathrm{UT}_n(R)$, in $g_1^{-1}(g_2^{-1} - e_n)$ and $g_1^{-1}g_2^{-1}(g_1 - e_n)$, the last rows must consist only of zeros. Hence the last row of h is $\mathbf{0}$. We have thus proved the inclusion $\gamma_1(\mathrm{UT}_{RB,\infty}(n, R)) \leqslant \mathrm{UT}_{RB,\infty}(n, 1, R)$.

Let $m \in \mathrm{UT}_{RB,\infty}(n, 1, R)$. Then

$$m = \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right),$$

for some $g \in \mathrm{UT}^1_n(R)$ and the last row of h consists of zeros. We need to show that h may be arbitrary. It suffices to put:

$$g_1 = \begin{pmatrix} 1 & 1 & 0 & \dots & \dots & 0 \\ & 1 & 1 & 0 & \dots & 0 \\ &&&&& \\ &&& 1 & 1 & 0 \\ &&&& 1 & 1 \\ &&&&& 1 \end{pmatrix}, \quad g_2 = e_n, \quad h' = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad h'' = \begin{pmatrix} 0 \\ (g_2 g_1 h)_1 \\ (g_2 g_1 h)_2 \\ \vdots \\ (g_2 g_1 h)_{n-1} \end{pmatrix}.$$

(We could also choose any other h' and the commutator $[m_1, m_2]$ would be the same.)

Suppose now that $\gamma_i(\mathrm{UT}_{Rb,\infty}(n, R)) = \mathrm{UT}_{RB,\infty}(n, i, R)$ for all $1 \leqslant i \leqslant k$. Let us consider $[\mathrm{UT}_{RB,\infty}(n, k, R), \mathrm{UT}_{RB,\infty}(n, R)]$. Again, it is easy to verify that if

$$m = \left(\begin{array}{c|c} g & h \\ \hline 0 & e \end{array}\right) \in \gamma_{k+1}(\mathrm{UT}_{RB,\infty}(n, R)),$$

then $g \in \mathrm{UT}^{k+1}_n(R)$, and conversely.

Let $m_1 \in \mathrm{UT}_{RB,\infty}(n, k, R)$, $m_2 \in \mathrm{UT}_{RB,\infty}(n, R)$ and h' be such that its last k rows are $\mathbf{0}$. Since $g_1 \in \mathrm{UT}^k_n(R)$, in $g_1 - e_n$ we have nonzero entries only above the kth diagonal. Then all elements of the last $k + 1$ rows of $g_1^{-1}g_2^{-1}(g_1 - e_n)h''$ must be equal to 0. For $j \geqslant 1$ only last $j - 1$ entries in jth row of $g_1^{-1}(g_2^{-1} - e_n)$ are nonzero; in $g_1^{-1}(g_2^{-1} - e_n)h'$ they are multiplied by elements from the last $n - j - 1$ rows of h', which are equal to $\mathbf{0}$. Hence the last $k + 1$ rows of h are $\mathbf{0}$ (this is depicted in Fig. 2).

To obtain arbitrary h we put:

$$g_1 = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots \\ & 1 & 0 & \dots & 0 & 1 & \\ && 1 & 0 &&& \\ &&& \ddots &&& \\ &&&&& 1 & 0 \\ &&&&& 0 & 1 \end{pmatrix}, \quad g_2 = e_n, \quad h' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad h'' = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (g_2 g_1 h)_1 \\ \vdots \\ (g_2 g_1 h)_{n-k-1} \end{pmatrix},$$

where g_1 is equal to $e + \sum_{i=1} n - k - 1 e_{i,i+k}$.

$$
\begin{pmatrix}
\overset{\leftarrow k+1 \rightarrow}{0 \; \cdots \; 0} & * & * & \cdots & * \\
 & & * & \cdots & * \\
 & & & \ddots & \vdots \\
 & & & & * \\
 & & & & 0 \\
 & & & & \vdots \\
 & & & & 0
\end{pmatrix}
\begin{pmatrix} * \\ * \\ \vdots \\ \vdots \\ * \\ * \\ 0 \end{pmatrix}
+
\begin{pmatrix}
0 & * & * & \cdots & & * \\
0 & 0 & * & & & \\
 & & & \ddots & & \vdots \\
 & & & & & \\
 & & & & * & * \\
 & & & & 0 & * \\
 & & & & 0 & 0
\end{pmatrix}
\begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\begin{matrix} \\ \\ \\ \uparrow \\ k \\ \downarrow \end{matrix}
$$

Fig. 2. Picture to the Proof of Theorem 4.3 – the lower central series of $\mathrm{UT}_{RB,\infty}(n, R)$.

Almost the same method is employed to prove the second claim. We discuss only the form of a matrix h. Suppose that $\mathrm{UT}^{(i)}_{RB,\infty}(n, R) = \mathrm{UT}_{RB,\infty}(n, 2^i - 1, R)$ for all $1 \leqslant i \leqslant k$. Let $m_1, m_2 \in \mathrm{UT}_{RB,\infty}(n, 2^k - 1, R)$. Then $[g_1, g_2] \in \mathrm{UT}_n^{2^{k+1}-1}(R)$. Matrices $g_1 - e_n$ and $e_n - g_2$ have, in jth row, the nonzero entries only in the last $n - j - 2^k + 1$ columns and h', h'' have nonzero entries only in firsts $n - 2^k + 1$ rows, so h may have nonzero entries only in the last $n - 2^{k+1} + 1$ rows. To obtain arbitrary h we put:

$$
g_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 1 & 0 & \ldots & & \\
 & 1 & 0 & \ldots & 0 & 1 & & & \\
 & & 1 & 0 & \ldots & 0 & 1 & & \\
 & & & 1 & 0 & & & & \\
 & & & & \ddots & & & & \\
 & & & & & & & 1 & 0 \\
 & & & & & & & 0 & 1
\end{pmatrix}, \quad g_2 = e_n, \quad h' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad h'' = \begin{pmatrix} 0 \\ \vdots \\ (g_2 g_1 h)_1 \\ \vdots \\ (g_2 g_1 h)_{n-2^{k+1}+1} \\ 0 \\ \vdots \end{pmatrix},
$$

where g_1 has all diagonals, except the main and 2^kth, equal to zero and the first 2^k rows of h'' are also equal to $\mathbf{0}$.

From what we have done above, it follows:

Corollary 4.1. *We have:*

1. *The group* $\mathrm{UT}_{RB,\infty}(n, R)$ *is nilpotent of class n.*
2. *The group* $\mathrm{UT}_{RB,\infty}(n, R)$ *is solvable of derived length* $\lceil \log_2(n + 1) \rceil$.

5. The derived and lower central series of $\mathbf{T}_{\boldsymbol{Rf},\infty}(\boldsymbol{R})$

For the purpose of this section we introduce few more notions. Infinite matrices of a form $s = \mathrm{Diag}(A_1, A_2, A_3, \ldots)$ are called *strings*. The infinite sequence $(\dim A_1, \dim A_2, \ldots)$ (consisting of dimensions of A_1, A_2 and so on) will be called a type of a string s defined as above. If for two strings s, t of types $(s_1, s_2, \ldots)$ and $(t_1, t_2, \ldots)$, the following condition is satisfied

$$\forall\, n \in \mathbb{N}\ \exists\, k, l \in \mathbb{N} \quad s_n = \sum_{i=k}^{l} t_i,$$

then we say that t is a substring of s. This is depicted in Fig. 3.

We present three properties of strings which will be needed.

Remark 5.1. If s is a string, then s^{-1} is a string and its type is the same as type of s.

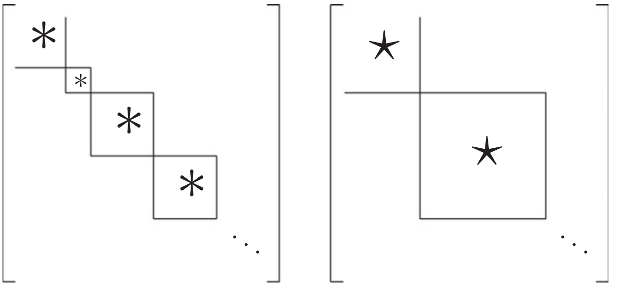

Fig. 3. In the picture the first string is a substring of the second one.

Remark 5.2. If s is a string and t is its substring, then the products st and ts are substrings of s.

Remark 5.3. Let $s \in \mathrm{UT}_{Rf,\infty}(m, R)$ be a string of a type $(s_1, s_2, \ldots)$. We define $s' = (s'_{ij})$ as follows

$$s'_{ij} = \begin{cases} a_{ij} & \text{for } \; j - i \leqslant m \\ 0 & \text{otherwise,} \end{cases}$$

where $a_{ij} \in R$.

Then s' is a substring of s.

We are going to use the following theorem (for the proof see [12]).

Theorem 5.1. *The group* $\mathrm{UT}_{Rf,\infty}(R)$ *is generated by strings. Every element of* $\mathrm{UT}_{Rf,\infty}(R)$ *is a string or a product of two strings.*

The claim of this theorem is also true for $\mathrm{T}_{Rf,\infty}(\infty, R)$. More information about this decomposition can be found in [13,14].

Here we prove few remarks and lemmata.

Remark 5.4. Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices from $\mathrm{UT}_n^k(R)$, where R is an arbitrary ring with the identity. If $AB \in \mathrm{UT}_n^{k+1}(R)$, then $b_{i,i+k} = -a_{i,i+k}$ for all $i = 1, 2, \ldots, n-k-1$.

Proof. We put $(c_{ij}) = C = AB$. Since $C \in \mathrm{UT}_n^{k+1}(R)$, we have $c_{i,i+k+1} = 0$ for all $i = 1, 2, \ldots, n - k - 1$. From $c_{i,i+k} = \sum_j a_{i,j}b_{j,i+k}$ and the fact that $a_{ij} = 0$ for $i < j \leqslant i + k$, we obtain $a_{ii}b_{i,i+k} + a_{i,i+k}b_{i+k,i+k} = 0$. Obviously $a_{ii} = b_{i+k,i+k} = 1$, so $b_{i,i+k} = -a_{i,i+k}$.

Remark 5.5. Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices from $\mathrm{UT}_{RF,\infty}(k, R)$, where R is an arbitrary ring with identity. If $AB \in \mathrm{UT}_{Rf,\infty}(k + 1, R)$, then we have $b_{i,i+k} = -a_{i,i+k}$ for all $i \in \mathbb{N}$.

The proof is analogous to the previous one.

Now using Theorem 5.1 and Remark 5.4 we prove the following.

Theorem 5.2. *Any matrix* $m \in \mathrm{UT}_{Rf,\infty}(k, R)$ *can be written as a product of at most two strings from* $\mathrm{UT}_{Rf,\infty}(k, R)$, *where* $k \in \mathbb{N}$.

Proof. Again, we prove by induction on k. Let $m \in \mathrm{UT}_{Rf,\infty}(1, R)$ be a product of two strings – $\hat{m}$ and $\tilde{m}$. From Remark 5.5 we know that $\tilde{m}_{ii+1} = -\hat{m}_{ii+1}$ for $i \in \mathbb{N}$. We put $a_1 = e + \sum_{i=1}^{\infty} \tilde{m}_{i,i+1}e_{i,i+1}$. The matrix a_1 is a string $\mathrm{Diag}(A_1^{(1)}, A_2^{(1)}, A_3^{(1)}, \ldots)$. We write m as $m = \hat{m}\tilde{m} = \hat{m}a_1a_1^{-1}\tilde{m}$. The matrices $\widehat{m_1} = \hat{m}a_1$, $\widetilde{m_1} = a_1^{-1}\tilde{m}$ are contained in $\mathrm{UT}_{Rf,\infty}(1, R)$. Moreover, from their definitions and

Remarks 5.1,5.2 and 5.3, we deduce that a_1 and a_1^{-1} are substrings of $\hat{m}$ and $\tilde{m}$. Hence, $\widehat{m_1}$ and $\widetilde{m_1}$ are substrings of $\hat{m}$, $\tilde{m}$, respectively, and they are contained in $\mathrm{UT}_{Rf,\infty}(1, R)$.

Suppose now that the claim holds for all $1 \leqslant i \leqslant k$. Let us consider $m \in \mathrm{UT}_{Rf,\infty}(k+1, R)$. By the inductive assumption, m can be written as a product of two strings $\hat{m}, \tilde{m} \in \mathrm{UT}_{Rf,\infty}(k, R)$. By Remark 5.5 we have $\tilde{m}_{ii+1} = -\hat{m}_{ii+1}$ for $i \in \mathbb{N}$. This time we put $a_k = e + \sum_{i=1}^{\infty} \tilde{m}_{ii+k} e_{ii+k}$. The matrix a_k is equal to a string $\mathrm{Diag}(A_1^{(k)}, A_2^{(k)}, A_3^{(k)}, \ldots)$. We have $m = \hat{m} a_k a_k^{-1} \tilde{m}$. We can see that $\widehat{m_k} = \hat{m} a_k$, $\widetilde{m_k} = a_k^{-1} \tilde{m}$ are contained in $\mathrm{UT}_{Rf,\infty}(k+1, R)$. Similarly to what we have done above, from Remarks 5.1, 5.2, 5.3 we obtain that a_k and a_k^{-1} are substrings of $\hat{m}$ and $\tilde{m}$ and so are $\widehat{m_k}$ and $\widetilde{m_k}$.

Now, one can see that it holds:

Lemma 5.1. *Every string $s \in \mathrm{UT}_{Rf,\infty}(R)$ can be written as a product of at most two commutators $c_1, c_2 \in \mathrm{T}_{Rf,\infty}(\infty, R)$.*

Proof. This proof is a corollary from Lemmata 3.3, 3.4. Let s be an unitriangular string equal to $\mathrm{Diag}(u_1, u_2, u_3, \ldots)$, where u_i are unitriangular finite matrices of dimensions $n(i)$ respectively. From Lemmata 3.3, 3.4 we know that every u_i can be written as a product $u_i = [c_i^{(1)}, c_i^{(2)}][c_i^{(3)}, c_i^{(4)}]$, where $c_i^{(j)} \in T_{n(i)}(R)$ for $1 \leqslant j \leqslant 4$, $i \in \mathbb{N}$. We put $c^{(j)} = \mathrm{Diag}(c_1^{(j)}, c_2^{(j)}, c_3^{(j)}, \ldots)$. Then $c_1 = [c^{(1)}, c^{(2)}]$ and $c_2 = [c^{(3)}, c^{(4)}]$ are desired commutators.

Proof of Theorems 1.3 and 1.4. The inclusion $\mathrm{T}'_{Rf,\infty}(R) \leqslant \mathrm{UT}_{Rf,\infty}(R)$ is obvious.

Let now $m \in \mathrm{UT}_{Rf,\infty}(R)$. If m is a string, then as it was shown in Lemma 5.1, it can be written as a product of at most 2 commutators. If m is a product of two strings m_1 and m_2, then (by Theorem 5.1) we are allowed to assume that they are unitriangular. In this case, by Lemma 5.1, both m_1 and m_2 can be written as products of at most 2 commutators, so our m is a product of at most 4 commutators.

The second statement can be proved the same way.

Theorem 5.3. *It holds:*

1. $\gamma_k(\mathrm{UT}_{Rf,\infty}(R)) = \mathrm{UT}_{Rf,\infty}(k, R)$.
2. $\mathrm{UT}_{Rf,\infty}^{(k)}(R) = \mathrm{UT}_{Rf,\infty}(2^k - 1, R)$.

Proof. We prove only the first point. At first we consider $k = 1$. It is easy to observe that $\mathrm{UT}'_{Rf,\infty}(R) \leqslant \mathrm{UT}_{Rf,\infty}(1, R)$. Let $m \in \mathrm{UT}_{Rf,\infty}(1, R)$. If m is a string, then the result follows. Otherwise, by Theorem 5.1, m can be written as a product $m = \hat{m}\tilde{m}$, where $\hat{m}, \tilde{m} \in UT_{Rf,\infty}(1, R)$. In this case the matrices $\hat{m}$ and $\tilde{m}$ are products of at most 2 commutators, so m is a product of at most 4 commutators. In a case when $k > 1$ the matrix m can be written as a product of two commutators (Lemma 3.3).

This confirms the known:

Corollary 5.1. *The group $\mathrm{UT}_{Rf,\infty}(R)$ is residually nilpotent.*

6. Conclusions

From previous sections it follows

Theorem 6.1. *The commutator of the subgroup of $\mathrm{GL}_{VK,\infty}(n, R)$ which consists of all matrices of the form*

$$\left(\begin{array}{c|c} g & h \\ \hline 0 & k \end{array}\right),$$

where $g \in \mathrm{GL}_n(R)$, $k \in \mathrm{UT}_{Rf,\infty}(R)$ *and* h *is arbitrary, consists of all matrices of the same form, but with* $g \in \mathrm{SL}_n(R)$*. Moreover, every element of the derived group may be written as a product of at most* 7 *commutators.*

Therefore we have

Theorem 6.2. *Consider the subgroup of Vershik–Kerov consisting of the elements of the form* $\left(\begin{array}{c|c} g & h \\ \hline 0 & k \end{array}\right)$, *where* k *is row-finite. Then the commutator subgroup consists of all matrices of the same form with* $g \in \mathrm{SL}_n(R)$*. Moreover, every element of the derived group may be written as a product of at most* 7 *commutators.*

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